THE CIRCULAR ELASTIC PLATE UNDER AN AXISYMMETRICAL TRANSVERSE LOAD

(KRUGLAIA UPRUGAIA PLITA PRI OSESIMMETRICHNOM POPERECHNOM NAGRUZHENII)

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I. G. TEREGULOV
(Kazan)

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Solutions are presented for two problems of the linear theory of elasticity, concerning the states of stress and deformation in circular elastic plates under symmetrical loading.

1. The equations of equilibrium of a thick isotropic elastic circular plate for symmetrical deformations under the action of transverse loads may be put in the form [1,2]

$$\sum_{k=0}^{\infty}(-1)^k\left\{\frac{1+k-\mu}{(2k)!}\frac{1}{a^2}\frac{d}{d\eta}\nabla^{2k-2}\frac{1}{\eta}\frac{d}{d\eta}\eta u_{(1)}+\frac{1-k-\mu}{(2k)!}\frac{1}{a^2}\frac{d}{d\eta}\nabla^{2k}w_{(0)}\right\}h^{2k+1}=0 \tag{1.1}$$

$$\sum_{k=0}^{\infty}(-1)^k\left\{\frac{1+k-\mu}{(2k+1)!}\nabla^{2k}\frac{1}{\eta}\frac{d}{d\eta}\eta u_{(1)}+\frac{1-k-\mu}{(2k+1)!}\nabla^{2k+2}w_{(0)}\right\}h^{2k+1}=\frac{1-\mu^2}{E}q$$

$$\sum_{k=0}^{\infty}(-1)^k\left\{\frac{k+\mu}{(2k+1)!}\frac{1}{a^2}\frac{d}{d\eta}\nabla^{2k}w_{(1)}+\frac{1+k-\mu}{(2k+1)!}\frac{1}{a^2}\frac{d}{d\eta}\nabla^{2k}\frac{1}{\eta}\frac{d}{d\eta}\eta u_{(0)}\right\}h^{2k+1}=0$$

$$\sum_{k=0}^{\infty}(-1)^k\left\{\frac{k-\mu}{(2k)!}\nabla^{2k}\frac{1}{\eta}\frac{d}{d\eta}\eta u_{(0)}\frac{k-1+\mu}{(2k)!}\nabla^{2k}w_{(1)}\right\}h^{2k+1}=\frac{(1+\mu)(1-2\mu)}{2E}p$$

Here $2h$ is the thickness, E the modulus of elasticity, μ the coefficient of transverse contraction (Poisson's ratio), a the radius of the plate, $0\leqslant\eta\leqslant 1$

$$q=-\pi_+-\pi_-,\qquad p=-h(\pi_+-h_-),\qquad \nabla^2(\ldots)=\frac{1}{a^2}\frac{1}{\eta}\frac{d}{d\eta}\eta\frac{d}{d\eta}(\ldots) \tag{1.2}$$

π_+, π_- are the normal external loads on $z=\pm h$

$$w=\sum_{k=0}^{\infty}w_{(k)}z^k,\qquad u=\sum_{k=0}^{\infty}u_{(k)}z^k,\qquad -h\leqslant z\leqslant h \tag{1.3}$$

w is the displacement normal to the middle plane, and u is the radial displacement.

Because of the relations [2]

$$w_{(2k)} = \frac{k(-1)^k}{2(2k)!(1-\mu)} \nabla^{2k-2} \frac{1}{\eta}\frac{d}{d\eta} \eta u_{(1)} - \frac{(k-2+2\mu)(-1)^k}{2(2k)!(1-\mu)} \nabla^{2k} w_{(0)} \tag{1.4}$$

$$w_{(2k+1)} = -\frac{(-1)^k}{(2k+1)!(1-2\mu)} \nabla^{2k} \frac{1}{\eta}\frac{d}{d\eta} \eta u_{(0)} - \frac{(k-1+2\mu)}{(2k+1)!(1-2\mu)} \nabla^{2k} w_{(1)}$$

$$u_{(2k)} = \frac{(-1)^k}{(2k)!(1-2\mu)} \frac{1}{a^2}\frac{d}{d\eta} \nabla^{2k-2} w_{(1)} + \frac{(k+1-2\mu)(-1)^k}{(2k)!(1-2\mu)} \frac{1}{a^2}\frac{d}{d\eta} \nabla^{2k-2} \frac{1}{\eta}\frac{d}{d\eta} \eta u_{(0)}$$

$$u_{(2k+1)} = -\frac{k(-1)^k}{2(2k+1)!(1-\mu)} \frac{1}{a^2}\frac{d}{d\eta} \nabla^{2k} w_{(0)} + \frac{(k+2-2\mu)(-1)^k}{2(2k+1)!(1-\mu)} \frac{1}{a^2}\frac{d}{d\eta} \nabla^{2k-2} \frac{1}{\eta}\frac{d}{d\eta} \eta u_{(1)}$$

the state of stress and deformation is determined by the four functions $w_{(0)}$, $u_{(0)}$, $w_{(1)}$, $u_{(1)}$. Let us assume that the loads may be represented by the series

$$b\,\frac{1-\mu^2}{E} = q_0 + \sum_{m=1}^{\infty} q_m \mu_m Z_0(\mu_m \eta), \qquad p\,\frac{(1+\mu)(1-2\mu)}{2Eh} = p_0 + \sum_{m=1}^{\infty} p_m Z_0(\mu_m \eta) \tag{1.5}$$

where $Z_n(\mu_m \eta)$ are functions satisfying the equation

$$\left(\frac{1}{\eta}\frac{d}{d\eta}\eta\frac{d}{d\eta} - \frac{n^2}{\eta^2}\right) Z_n = -\mu_m^2 Z_n \tag{1.6}$$

and the numbers μ_m, which are arranged in increasing order, are the non-zero roots of the equation

$$Z_1(x) = 0, \qquad \mu_{m+1} > \mu_m \geqslant \mu_1 > 0 \tag{1.7}$$

We will select the specific functions Z_n in accordance with the character of the particular problem. Clearly

$$q_m = \frac{1-\mu^2}{N_m \mu_m E} \int_\lambda^1 q\eta Z_0(\mu_m \eta)\, d\eta, \qquad p_m = \frac{(1+\mu)(1-2\mu)}{2EhN_m} \int_\lambda^1 p\eta Z_0(\mu_m \eta)\, d\eta \tag{1.8}$$

$$q_0 = \frac{2(1-\mu^2)}{E} \int_\lambda^1 q\eta\, d\eta, \qquad p_0 = \frac{(1+\mu)(1-2\mu)}{Eh} \int_\lambda^1 p\eta\, d\eta \qquad \left(N_m = \int_\lambda^1 \eta Z_0^2(\mu_m \eta)\, d\eta\right)$$

since the system of functions $Z_0(\mu_m \eta)$ is orthogonal on the interval $\lambda \leqslant \eta \leqslant 1$. Here $\lambda = 0$ corresponds to a plate without a center hole, while $\lambda > 0$ corresponds to the case of a thick ring, the inner central cylindrical hole having radius λa.

We consider the case in which the plate is supported in such a manner that the vertical deflection on the circle $\eta = \eta_0 (\lambda \leqslant \eta_0 \leqslant 1)$ is prevented by reactions on $z = -h$, distributed around the circumference of this circle. The total reaction equals

$$R = 2\pi a^2 \int_\lambda^1 (\pi_+ + \pi_-)\, \eta d\eta$$

Assuming that this reaction is uniformly distributed on an area of the plane $z = -h$ bounded by the circles $\eta_0 + \delta/2$ and $\eta_0 - \delta/2$, we obtain

$$q = -\pi_+ - \pi_- + \frac{1}{\eta_0 \delta} \int_\lambda^1 (\pi_+ + \pi_-)\, U\left(\eta_0 \pm \frac{\delta}{2}\right) \eta d\eta \tag{1.9}$$

$$p = -h(\pi_+ - \pi_-) - \frac{1}{\eta_0 \delta} \int_\lambda^1 (\pi_+ + \pi_-)\, U\left(\eta_0 \pm \frac{\delta}{2}\right) \eta d\eta$$

where

$$U\left(\eta_0 \pm \frac{\delta}{2}\right) = \begin{cases} 0 \text{ for } \eta < \eta_0 - \frac{1}{2}\delta, \quad \eta > \eta_0 + \frac{1}{2}\delta \\ 1 \quad \text{for } \eta_0 - \frac{1}{2}\delta \leqslant \eta \leqslant \eta_0 + \frac{1}{2}\delta \end{cases}$$

In accordance with these expressions, we obtain for p_m and q_m

$$q_m = -\frac{1-\mu^2}{EN_m \mu_m} \int_\lambda^1 (\pi_+ + \pi_-)\, [Z_0(\mu_m \eta) - Z_0(\mu_m \eta_0)]\, \eta d\eta$$

$$p_m = -\frac{(1+\mu)(1-2\mu)}{2EN_m} \int_\lambda^1 [(\pi_+ - \pi_-)\, Z_0(\mu_m \eta) + (\pi_+ + \pi_-)\, Z_0(\mu_m \eta_0)]\, \eta d\eta \tag{1.10}$$

$$q_0 = 0, \qquad p_0 = -\frac{2(1+\mu)(1+2\mu)}{E} \int_\lambda^1 \pi_+ \eta d\eta$$

where the limiting process $\delta \to 0$ has been carried out.

We seek a solution of the system (1.1) in the form

$$w_{(0)} = \alpha_1 + \alpha_2 \ln \eta + \alpha_3 \eta^2 + \alpha_4 \eta^2 \ln \eta + \alpha_5 \eta^4 + a \sum_{m=1}^{\infty} A_m Z_0(\mu_m \eta)$$

$$u_{(1)} = \beta_1 \frac{1}{\eta} + \beta_2 \eta + \beta_3 \eta \ln \eta + \beta_4 \eta^3 + \frac{1}{a} \sum_{m=1}^{\infty} B_m \mu_m Z_1(\mu_m \eta) \tag{1.11}$$

$$w_{(1)} = \gamma + \sum_{m=1}^{\infty} C_m Z_0(\mu_m \eta), \qquad u_{(0)} = \theta_1 \eta + \frac{\theta_2}{\eta} + \sum_{m=1}^{\infty} \frac{1}{\mu_m} D_m Z_1(\mu_m \eta)$$

After substituting these expressions for $w_{(0)}, \ldots, u_{(0)}$ into Equations (1.1) and making use of (1.5) and (1.10), we find that in order to satisfy these equations the following conditions must be fulfilled:

$$\beta_1 = -\frac{\alpha_2}{a^2} - \frac{4h^2}{(1-\mu)a^2}\alpha_4, \qquad \beta_2 = -\frac{2\alpha_3}{a^2} - \frac{\alpha_4}{a^2} \tag{1.12}$$
$$\beta_3 = -\frac{2\alpha_4}{a^2}, \qquad \beta_4 = 0, \qquad \alpha_5 = 0, \qquad \gamma = -\frac{p_0}{1-\mu} - \frac{2\mu}{1-\mu}\theta_1$$

We obtain from Equations (1.1) a system of linear equations for the determination of A_m, B_m, C_m and D_m, which yield

$$A_m = \frac{q_m}{1-\mu}\frac{\zeta_m \sinh\zeta_m + 2(1-\mu)\cosh\zeta_m}{2\zeta_m - \sinh 2\zeta_m}, \qquad B_m = -\frac{q_m}{1-\mu}\frac{\zeta_m \sinh\zeta_m - 2(1-\mu)\cosh\zeta_m}{2\zeta_m - \sinh 2\zeta_m} \tag{1.13}$$
$$C_m = -\frac{2p_m}{1-2\mu}\frac{\zeta_m \cosh\zeta_m + (1-2\mu)\sinh\zeta_m}{2\zeta_m + \sinh 2\zeta_m}, \qquad D_m = \frac{2p_m}{1-2\mu}\frac{\zeta_m \cosh\zeta_m - (1-2\mu)\cosh\zeta_m}{2\zeta_m + \sinh 2\zeta_m}$$
$$\zeta_m = \mu_m \frac{n}{a}$$

Making use of (1.3), (1.4), (1.11), (1.12) and (1.13), we obtain

$$w = \alpha_1 + \alpha_2 \ln\eta + \alpha_3\eta^2 + \alpha_4\eta^2\ln\eta - z\left(\frac{p_0}{1-\mu} + \frac{2\mu}{1-\mu}\theta_1\right) + \frac{z^2\mu}{2(1-\mu)a^2} \times \tag{1.14}$$
$$\times\,(\alpha_3 + 8\alpha_4\ln\eta + 4\alpha_4) + \frac{a}{1-\mu}\sum_{m=1}^{\infty} q_m X_m Z_0(\mu_m\eta) + \frac{2z}{1-2\mu}\sum_{m=1}^{\infty} p_m L_m Z_0(\mu_m\eta)$$
$$u = \theta_1\eta + \frac{\theta_2}{\eta} - z\left\{\frac{1}{\eta}\left[\frac{\alpha_2}{a^2} + \frac{4h^2}{(1-\mu^2)a^2}\right] + \eta\left(\frac{2\alpha_3}{a^2} + \frac{\alpha_4}{a^2}\right) + \frac{2\alpha_4}{a^2}\eta\ln\eta\right\} +$$
$$+\, z^3\frac{2(2-\mu)}{3(1-\mu)a^4}\frac{\alpha_4}{\eta} + \frac{1}{1-\mu}\sum_{m=1}^{\infty} q_m Y_m Z_1(\mu_m\eta) + \frac{2}{1-2\mu}\sum_{m=1}^{\infty} p_m M_m Z_1(\mu_m\eta)$$

where

$$X_m = \frac{\zeta_m \sinh\zeta_m \cosh\gamma_m + \gamma_m \sinh\zeta_m \cosh\gamma_m + 2(1-\mu)\cosh\zeta_m \cosh\gamma_m}{2\zeta_m - \sinh 2\zeta_m}$$
$$Y_m = \frac{\gamma_m \cosh\gamma_m \cosh\zeta_m - \zeta_m \sinh\zeta_m \sinh\gamma_m + (1-2\mu)\cosh\zeta_m \sinh\gamma_m}{2\zeta_m - \sinh 2\zeta_m}$$
$$L_m = \frac{\gamma_m \cosh\gamma_m \sinh\zeta_m - \zeta_m \cosh\zeta_m \sinh\gamma_m - 2(1-\mu)\sinh\gamma_m \sinh\zeta_m}{\gamma_m(2\zeta_m + \sinh 2\zeta_m)} \tag{1.15}$$
$$M_m = \frac{\zeta_m \cosh\zeta_m \cosh\gamma_m - \gamma_m \sinh\gamma_m \sinh\zeta_m - (1-2\mu)\cosh\gamma_m \sinh\zeta_m}{\mu_m(2\zeta_m + \sinh 2\zeta_m)}$$
$$\gamma_m = \mu_m \frac{z}{a}$$

2. We now consider the problem of the state of stress and deformation in a thick circular plate without a center hole, when the points on the bounding cross-section are restrained from moving in the radial direction and the shearing stress $\tau_{z\rho}$ is equal to zero at each point of this boundary (for $\eta = 1$). These conditions will be satisfied if

$$\begin{aligned} u_{(n)} &= 0 \quad \text{for } \eta = 1, \quad -h \leqslant z \leqslant h, \ n = 0, 1, 2, \ldots \\ \frac{dw_{(n)}}{d\eta} &= 0 \quad \text{for } \eta = 1, \quad -h \leqslant z \leqslant h, \ n = 0, 1, 2, \ldots \\ w &= 0 \quad \text{for } \eta = \eta_0, \quad z = -h \end{aligned} \tag{2.1}$$

Since the plate has no hole

$$\alpha_2 = \alpha_4 = \beta_1 = \beta_3 = \theta_2 = 0 \tag{2.2}$$

in view of the fact that the displacements must be bounded at the point $\eta = 0$, $z = 0$.

If we take

$$Z_n(x) = J_n(x) \tag{2.3}$$

where J_n are Bessel functions of the first kind of order n, which are orthogonal on the interval $0 \leqslant \eta \leqslant 1$, then in view of (1.7) we conclude that the terms containing A_m, B_m, C_m and D_m in the expressions for the displacements automatically satisfy the conditions (2.1). For the remaining constants we obtain from conditions (2.1) and (1.12)

$$\theta_1 = 0, \quad \alpha_3 = 0, \quad \gamma = -\frac{p_0}{1-\mu}$$

The constant α_1 is determined from condition (2.1), but it has no effect on the states of stress and deformation, hence its determination is omitted.

3. We consider the problem of the states of stress and deformation in a thick annular plate, the boundaries of which are cylinders of radii a and λa $(0 < \lambda < 1)$. On these cross-sections the radial displacement and shearing stress $\tau_{z\rho}$ are zero. These boundary conditions will be satisfied if

$$\begin{aligned} u_{(n)} &= 0 \quad \text{for } \eta = 1, \quad \eta = \lambda, \quad n = 0, 1, 2, \ldots \\ \frac{dw_{(n)}}{d\eta} &= 0 \quad \text{for } \eta = 1, \quad \eta = \lambda, \quad n = 0, 1, 2, \ldots \\ w &= 0 \quad \text{for } \eta = \eta_0, \quad z = -h \quad (\lambda \leqslant \eta_0 \leqslant 1) \end{aligned} \tag{3.1}$$

For this case we obtain

$$Z_n(\mu_m\eta, \lambda) = S_n(\mu_m\eta, \lambda) = J_n(\mu_m\eta) Y_1(\mu_m\lambda) - J_1(\mu_m\lambda) Y_n(\mu_m\eta) \tag{3.2}$$

where $Y_n(x)$ is the Bessel function of the second kind of order n. If the numbers μ_m are the roots of Equation (1.7), or $S_1(x, \lambda) = 0$, then we have the equalities

$$S_1(\mu_m, \lambda) = 0, \quad S_1(\mu_m\lambda, \lambda) = 0 \tag{3.3}$$

The system of functions $S_0(\mu_m\eta, \lambda)$ is orthogonal on the interval $\lambda \leqslant \eta \leqslant 1$. For such a choice of the functions Z_n the terms containing A_m, B_m, C_m and D_m in the expressions for the displacements automatically satisfy (3.1). We obtain for the remaining constants

$$\theta_1 = \theta_2 = \alpha_2 = \alpha_3 = \alpha_4 = 0, \qquad \gamma_1 = -\frac{p_0}{1-\mu} \tag{3.5}$$

4. In order to obtain numerical results one must know the roots of Equation (1.7) for the function (2.3) and the roots of Equations (3.3). The roots of the first of these equations may be found, for example, in [3]. The asymptotic expansions of the functions $J_1(x)$ and $Y_1(x)$ may be used to advantage in determining the roots of the second equation, and one finds

$$y_{i+1} = \pi n + \tan^{-1} \frac{Q_1\left(\frac{\lambda y_i}{1-\lambda}\right) P_1\left(\frac{y_i}{1-\lambda}\right) - Q_1\left(\frac{y_i}{1-\lambda}\right) P_1\left(\frac{\lambda y_i}{1-\lambda}\right)}{P_1\left(\frac{y_i}{1-\lambda}\right) P_1\left(\frac{\lambda y_i}{1-\lambda}\right) + Q_1\left(\frac{y_i}{1-\lambda}\right) Q_1\left(\frac{\lambda y_i}{1-\lambda}\right)} \tag{4.1}$$

Here $y = x(1-\lambda)$, $y_1 = \pi n$, and x is a root of Equations (3.3); in order to obtain the necessary accuracy one should use successive approximations, which converge rapidly. The functions $Q_1(x)$ and $P_1(x)$ have the form

$$Q_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (4k-1)!!\,(4k+3)!!}{(2k+1)!\,(8x)^{2k+1}} \approx \frac{0.375}{x} - \frac{0.1025}{x^3}$$
$$P_1(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (4k-3)!!\,(4k+1)!!}{(2k)!\,(8x)^{2k}} + 1 \approx 1 + \frac{0.1171}{x^2} \tag{4.2}$$

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